SHARP BOUND FOR THE ERGODIC MAXIMAL OPERATOR ASSOCIATED TO CESÀRO BOUNDED OPERATORS

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Abstract. We consider positive invertible Lamperti operators $Tf(x) = h(x)\Phi f(x)$ such that Φ has no periodic part. Let $A_{n,T}$ be the sequence of averages of T and M_T the ergodic maximal operator. It is obvious that if M_T is bounded on some L^p, $1 < p < \infty$, then $\sup ||A_{n,T}||_{L^p(\nu)} \le ||M_T||_{L^p(\nu)} < \infty$. It is known that the converse is true. In this paper we search the sharp dependence of the norm $||M_T||_{L^p(\nu)}$ with respect to $\sup_n ||A_{n,T}||_{L^p(\nu)} < \infty$. We prove that $\|M_T\|_{L^p(\nu)} \le C(p)(\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(d\nu)})^{p'}$, where $p' = p/(p-1)$ is the conjugate exponent and $C(p)$ depends only on p. Furthermore, the exponent p' is sharp. Our results are closely related to Buckley's theorem about sharp bounds for the Hardy-Littlewood maximal function.

1. Introduction

Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and let $\mathcal{M}(\mu)$ be the space of measurable functions $f : (X, \mathcal{F}) \to \mathbb{R}$ where, as usual, we identify functions which are equal almost everywhere. By $L^p := L^p(\mu)$, $1 \le p < \infty$, we denote the measurable functions f such that $\int_X |f|^p \, d\mu < \infty$. For $f \in L^p$, we write $||f||_p = ||f||_{L^p(d\mu)} = \left(\int_X |f|^p \, d\mu\right)^{1/p}$.

Associated to a linear operator $T : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ (or alternatively $T : L^p(\mu) \to L^p(\mu)$), we consider the sequence $A_{n,T} : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ of operators (averages) defined

2010 *Mathematics Subject Classification.* 47A35, 37A40, 37A45, 28D05, 42B25.
Key words and phrases. Cesàro bounded operators, Mean bounded operators, Lamperti operators, ergodic maximal operator, Sharp bound, one-sided weights.
The first author is partially supported by grants from Agencia Nacional de Promoción Científica y Tecnológica (ANPCyT) and Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET). The second author is supported by grants MTM2011-28149-C02-02 and MTM2015-66157-C2-2-P of the Ministerio de Economía y Competitividad (Spain) and grant FQM-354 of the Junta de Andalucía.

by

$$A_{n,T}f = \frac{1}{n+1}\sum_{j=0}^{n} T^j f, \tag{1.1}$$

and the ergodic maximal operator

$$M_T f = \sup_{n\geq 0} |A_{n,T}f|. \tag{1.2}$$

Akcoglu's theorem [1] says that if $1 < p < \infty$ and T is a positive linear contraction on L^p then

$$||M_T f||_p \leq \frac{p}{p-1}||f||_p, \tag{1.3}$$

and the sequence of averages $A_{n,T}f$ converges a.e. and in the norm of L^p for all $f \in L^p$ (we recall that positive means that if $f \geq 0$ a.e then $Tf \geq 0$ a.e and contraction stands for $||T|| \leq 1$). As usual, the norm of M_T, denoted by $||M_T||$ or $||M_T||_p$, is defined as the least constant C_p such that $||M_T f||_p \leq C_p||f||_p$ for all $f \in L^p$. Thus, the above inequality says that $||M_T||_p \leq \frac{p}{p-1}$ for all positive linear contractions T on L^p.

The proof of Akcoglu's theorem follows from the particular case of positive isometries (T is a positive linear operator and $||T|| = 1$) which was previously proved by A. Ionescu-Tulcea [2]. The proof of Ionescu-Tulcea's result in Krengel's book [4] follows the lines of the proofs by Kan [3] and de la Torre [12]. It is based on the following key fact: if $1 < p < \infty$ and T is a positive linear isometry on L^p then T is a Lamperti operator or, in other words, T separates supports ($fg = 0$ a.e. $\Rightarrow TfTg = 0$ a.e.). As a first question we may wonder whether or not $p/(p-1)$ is the best constant in inequality (1.3) for positive invertible linear isometries on L^p. We answer to this question in the affirmative in Section 6 for positive linear isometries such that its associated automorphism has no periodic part (see Definition 2.1); obviously, the answer is negative for trivial cases like the identity). This result is probably known but we have not found any reference.

As we have noticed, Lamperti operators are a very important case. For that reason, we choose these kind of operators as the setting in the paper. Lamperti operators have a very special structure [3, 5] that we resume in Section 2.

In [10] (see also the previous paper [7]) it was proved a kind of generalization of Akcoglu's theorem. On the one hand, more restrictive assumptions are considered: the author works with positive invertible Lamperti operators and a measure $\nu = w\, d\mu$ where w is a nonnegative measurable function. On the other hand, the author treats with an assumption more general : he does not assume that T is a positive contraction but the averages are uniformly bounded in $L^p(\nu)$, that is

$$\sup_n ||A_{n,T}||_{L^p(\nu)} < \infty$$

and, under these assumptions, it is proved that the maximal operator M_T is bounded in $L^p(\nu)$. It is clear that $\sup_n ||A_{n,T}||_{L^p(\nu)} \leq ||M_T||_{L^p(\nu)}$. In this paper we search the sharp dependence of the norm $||M_T||_{L^p(\nu)}$ with respect to $\sup_n ||A_{n,T}||_{L^p(\nu)} < \infty$. We establish that if the associated automorphism has no periodic part then

$$\|M_T\|_{L^p(\nu)} \leq C(p)(\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(d\nu)})^{p'}, \tag{1.4}$$

where $p' = p/(p-1)$ is the conjugate exponent and $C(p)$ depends only on p. Furthermore, the exponent is sharp (see Theorems 3.1 and 3.2).

The paper is organized in the following way: Section 2 is devoted to establish the setting of the paper; in particular we resume the structure and properties of Lamperti operators. The next section contains the main results and the proofs of the results are in the following sections.

2. Lamperti operators

In this section we state the setting of our paper (which is the same as in [10]). A Lamperti operator on $\mathcal{M}(\mu)$ is a map $T : \mathcal{M}(\mu) \to \mathcal{M}(\mu)$ of the form

$$Tf(x) = h(x)\Phi f(x), \tag{2.1}$$

where $h \in \mathcal{M}(\mu)$ and $\Phi : \mathcal{M}(\mu) \longrightarrow \mathcal{M}(\mu)$ is linear and multiplicative, that is,

(1) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$
(2) $\Phi(fg) = \Phi(f)\Phi(g)$

Throughout the paper we always assume that T is positive and invertible. It follows that $0 < h(x) < \infty$ a.e. and Φ is invertible and positive. Other properties are $\Phi 1 = 1$, $\Phi(|f|^r) = |\Phi(f)|^r$ for positive r and the following ones (see e.g. [3] y [5]):

(1) There exists a sequence of functions h_j such that

$$T^j f = h_j \Phi^j f \tag{2.2}$$

where $h_1 = h$, $h_0 = 1$ and $h_{j+k} = h_j\, \Phi^j h_k$, for any j, k in $\mathbb{Z}$.

(2) By the Radon-Nikodym theorem, for every $j \in \mathbb{Z}$ there exists a positive function $J_j \in \mathcal{M}(\mu)$ such that if $f \geq 0$ then

$$\int_X J_j\, \Phi^j f\, d\mu = \int_X f\, d\mu \qquad \text{and} \qquad J_{j+k} = J_j\, \Phi^j J_k. \tag{2.3}$$

We finish this section with one definition which plays an important role in the results of this paper.

Definition 2.1. *If Φ is as before, we say that Φ is aperiodic or, in other words, it has no periodic part if for any $n \geq 1$ and $E \subset \mathcal{F}$ with $\mu(E) > 0$ there exists a non-null measurable subset A of E such that $\Phi^n \chi_A \neq \chi_A$.*

Given any bimeasurable measure preserving transformation $\tau : X \to X$ we consider $\Phi f(x) = f(\tau(x))$. The morphism Φ is aperiodic if τ is ergodic and $\mu(X) = \infty$ or τ is ergodic and $(X, \mathcal{F}, \mu)$ is a finite nonatomic measure space. An example of an aperiodic Φ such that τ is not ergodic is the one induced by $\tau : [0,1] \times [0,1]$, $\tau(x,y) = ((x+a)\, mod\, 1, y)$, where a is irrational (see [11]).

3. Statement of the main results

A Cesàro bounded operator in $L^p(wd\mu)$ is a linear operator such that the averages are uniformly bounded in $L^p(wd\mu)$, that is, $\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)} < \infty$. Under this assumption the next theorem estimates the norm of the maximal operator associated to a positive invertible Lamperti operator $Tf(x) = h(x)\Phi f(x)$ when Φ has no periodic part.

Theorem 3.1. *Let $Tf(x) = h(x)\Phi f(x)$ a positive invertible Lamperti operator such that Φ has no periodic part. Let w be a nonnegative measurable function on X and let $1 < p < \infty$. If T is Cesàro bounded operator in $L^p(wd\mu)$ then the maximal operator M_T is bounded in $L^p(wd\mu)$ and*

$$||M_T||_{L^p(wd\mu)} \le C(p) \left(\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)} \right)^{p'},$$

where $C(p)$ depends only on p.

The second theorem establishes that the above inequality is sharp.

Theorem 3.2. *Let $\Phi : \mathcal{M}(\mu) \longrightarrow \mathcal{M}(\mu)$ invertible, linear and multiplicative and such that Φ has no periodic part. Assume that there exist p_0, $1 < p_0 < \infty$, a constant $\beta > 0$ and a constant $C(p_0)$ depending only on p_0 such that*

$$||M_T||_{L^{p_0}(wd\mu)} \le C(p_0) \left(\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^{p_0}(wd\mu)} \right)^{\beta}$$

for all nonnegative measurable functions w on X and all positive invertible Lamperti operators $Tf = h\,\Phi f$. Then $\beta \geq p_0'$.

In order to prove the first theorem we need to compute the norm of the averages $A_{n,T}$. This is included in the next result.

Theorem 3.3. *Let w be a nonnegative measurable function on X. Let $Tf = h\,\Phi f$ a positive invertible Lamperti operator on $\mathcal{M}(\mu)$ such that it has no periodic part. Let $1 < p < \infty$. The following statements are equivalent.*

(a) *T is a Cesàro bounded operator in $L^p(wd\mu)$.*

(b) *$w \in A_p^+(T)$, i.e., there exists a positive constant C such that for a.e. $x \in X$ and all $k \in \mathbb{N}$*

$$\left(\sum_{i=-k}^{0} h_i^{-p}(x)J_i(x)\Phi^i w(x)\right)\left(\sum_{i=0}^{k}[h_i^{-p}(x)J_i(x)\Phi^i w(x)]^{\frac{-1}{p-1}}\right)^{p-1} \leq C(k+1)^p. \tag{3.1}$$

Furthermore, if $[w]_{A_p^+(T)}$ stands for the infimum of the constants in (3.1) then we have

$$\frac{1}{2}[w]_{A_p^+(T)}^{1/p} \leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)} \leq 4[w]_{A_p^+(T)}^{1/p}. \tag{3.2}$$

Remark 3.4. *Inequality 3.1 must be understood in the following way: if $\Phi^i w(x) = 0$ for some i, $0 \leq i \leq k$, then $\Phi^j w(x) = 0$ for all j such that $-k \leq j \leq 0$; if $\Phi^i w(x) = \infty$ then $[\Phi^i w(x)]^{\frac{-1}{p-1}} = 0$; if $\Phi^i w(x) = \infty$ for some i, $-k \leq i \leq 0$, then $\Phi^i w(x) = \infty$ for all i, $0 \leq i \leq k$. Similar conditions appearing in this paper must be understood in the same way.*

Remark 3.5. *$w \in A_p^+(T)$ if and only if there exists a positive constant C such that for a.e. $x \in X$ all integers j and all $k \in \mathbb{N}$*

$$\left(\sum_{i=j-k}^{j} h_i^{-p}(x)J_i(x)\Phi^i w(x)\right)\left(\sum_{i=j}^{j+k}[h_i^{-p}(x)J_i(x)\Phi^i w(x)]^{\frac{-1}{p-1}}\right)^{p-1} \leq C(k+1)^p.$$

Notice that the infimum of the constants in the above inequality equals $[w]_{A_p^+(T)}$.

In the proof of Theorem 3.2 we need to compute the norm of the maximal operator associated to a positive invertible isometry. This result is probably known but we have not found any reference. We include a proof to make the article more self-contained.

Theorem 3.6. *Let $1 < p < \infty$. Let T_p be a positive invertible Lamperti operator $T_p f = h\,\Phi f$ which is an isometry on $L^p(\mu)$, that is,*

$$T_p f(x) = J_1(x)^{1/p}\Phi f(x).$$

Assume that Φ has no periodic part. Then

$$\|M_{T_p}\|_{L^p(d\mu)} = \frac{p}{p-1}.$$

4. Proof of Theorem 3.3

Proof. Let's start by proving that if (b) holds then T is a Cesàro bounded operator in $L^p(wd\mu)$ and

$$\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)} \leq 4[w]^{1/p}_{A_p^+(T)}.$$

We consider first the averages

$$\tilde{A}_{2^k} f(x) = \frac{1}{2^k}\sum_{i=2^k}^{2^{k+1}-1} T^i f(x).$$

and we prove that

$$\|\tilde{A}_{2^k}\|_{L^p(wd\mu)} \leq 2[w]^{1/p}_{A_p^+(T)}$$

for all $k \geq 0$.

We may assume that the functions f are nonnegative. Let $u_i(x) = h_i^{-p}(x)J_i(x)\Phi^i w(x)$. Notice that by Remark 3.4, if $A = \{x : u_i(x) = 0 \text{ for some } i,\ 2^k \leq i \leq 2^{k+1}-1\}$, then $w(x) = 0$ for a.e. $x \in A$. We also point out that if $B = \{x : u_i(x) = \infty\}$ then for all f in $L^p(wd\mu)$ we have that $\Phi^i f(x) = 0$ for a.e. $x \in B$.

Using that Φ is linear and multiplicative, identities (2.2) and (2.3), Hölder's inequality and what we have pointed out before, we have

(4.1)

$$\begin{aligned}
\|\tilde{A}_{2^k} f\|^p_{L^p(w d\mu)} &= \int_X \left| \frac{1}{2^k} \sum_{i=2^k}^{2^{k+1}-1} T^i f \right|^p w \, d\mu = \int_X \left| \frac{1}{2^k} \sum_{i=2^k}^{2^{k+1}-1} h_i(x) \Phi^i f \, u_i^{1/p} u_i^{-1/p} \right|^p w \, d\mu \\
&\le \frac{1}{2^{kp}} \int_X \left(\sum_{i=2^k}^{2^{k+1}-1} (h_i \Phi^i f)^p u_i \right) \left(\sum_{j=2^k}^{2^{k+1}-1} u_j^{1-p'} \right)^{p-1} w \, d\mu \\
&= \frac{1}{2^{kp}} \sum_{i=2^k}^{2^{k+1}-1} \int_X J_{-i} \, \Phi^{-i}(h_i^p) \Phi^{-i}(\Phi^i f^p) \, \Phi^{-i}(u_i) \\
&\qquad \times \left(\sum_{j=2^k}^{2^{k+1}-1} (\Phi^{-i} u_j)^{1-p'} \right)^{p-1} \Phi^{-i} w \, d\mu \\
&= \frac{1}{2^{kp}} \sum_{i=2^k}^{2^{k+1}-1} \int_X f^p \, w \left(\sum_{j=2^k}^{2^{k+1}-1} (\Phi^{-i} u_j)^{1-p'} \right)^{p-1} \Phi^{-i} w \, d\mu.
\end{aligned}$$

If we use (2.2) and (2.3) again then we obtain

$$\begin{aligned}
&\left(\sum_{j=2^k}^{2^{k+1}-1} (\Phi^{-i} u_j)^{1-p'} \right)^{p-1} \\
&\qquad = \left(\sum_{j=2^k}^{2^{k+1}-1} \left[\Phi^{-i}(h_j^{-p}) \Phi^{-i}(J_j) \Phi^{-i+j} w \right]^{1-p'} \right)^{p-1} \\
&\qquad = J_{-i} h_{-i}^{-p} \left(\sum_{j=2^k}^{2^{k+1}-1} \left[h_{-i}^{-p} \Phi^{-i}(h_j^{-p}) \, J_{-i} \Phi^{-i}(J_j) \, \Phi^{-i+j} w \right]^{1-p'} \right)^{p-1} \\
&\qquad = J_{-i} h_{-i}^{-p} \left(\sum_{j=2^k}^{2^{k+1}-1} \left[h_{-i+j}^{-p} \, J_{-i+j} \, \Phi^{-i+j} w \right]^{1-p'} \right)^{p-1} \\
&\qquad = J_{-i} h_{-i}^{-p} \left(\sum_{j=2^k}^{2^{k+1}-1} u_{-i+j}^{1-p'} \right)^{p-1}.
\end{aligned}$$

Putting the last equality in (4.1) and taking into account that $-2^k+1 \leq -i+j \leq 2^k-1$, we get

$$\begin{aligned}
\|\tilde{A}_{2^k}\|^p_{L^p(wd\mu)} &\leq \frac{1}{2^{kp}} \sum_{i=2^k}^{2^{k+1}-1} \int_X f^p\, w\; J_{-i} h_{-i}^{-p} \Phi^{-i} w \left(\sum_{j=2^k}^{2^{k+1}-1} u_{-i+j}^{1-p'} \right)^{p-1} d\mu \\
&= \frac{1}{2^{kp}} \int_X f^p\, w \left(\sum_{i=2^k}^{2^{k+1}-1} u_{-i} \left(\sum_{j=2^k}^{2^{k+1}-1} u_{-i+j}^{1-p'} \right)^{p-1} \right) d\mu \\
&\leq \frac{1}{2^{kp}} \int_X f^p\, w \left(\sum_{i=2^k}^{2^{k+1}-1} u_{-i} \right) \left(\sum_{l=-2^k+1}^{2^k-1} u_l^{1-p'} \right)^{p-1} d\mu \\
&= \frac{1}{2^{kp}} \int_X f^p\, w \left(\sum_{l=-2^{k+1}+1}^{-2^k} u_l \right) \left(\sum_{l=-2^k+1}^{2^k-1} u_l^{1-p'} \right)^{p-1} d\mu \\
&\leq \frac{1}{2^{kp}} \int_X f^p\, w \left(\sum_{l=-2^{k+1}+1}^{-2^k} u_l \right) \left(\sum_{l=-2^k}^{2^k-1} u_l^{1-p'} \right)^{p-1} d\mu \\
&\leq \frac{2^{(k+1)p}}{2^{kp}} [w]_{A_p^+(T)} \int_X f^p\, w\, d\mu \\
&= 2^p [w]_{A_p^+(T)} \int_X f^p\, w\, d\mu,
\end{aligned}$$

as we wished to prove.

Now we compare the general averages $A_{n,T}$ with $\tilde{A}_{2^k}$. Since $A_{0,T}f(x) = f(x)$, it is enough to consider $n \geq 1$. In such a case, there exists $j \in \mathbb{N}$ such that $2^j \leq n \leq 2^{j+1}-1$. Then we have

$$\begin{aligned}
A_{n,T}f(x) = \frac{1}{n+1} \sum_{i=0}^{n} T^i f(x) &\leq \frac{1}{n+1} \sum_{i=0}^{2^{j+1}-1} T^i f(x) = \frac{1}{n+1} \left(f(x) + \sum_{i=1}^{2^{j+1}-1} T^i f(x) \right) \\
&= \frac{1}{n+1} \left(f(x) + \sum_{k=0}^{j} \sum_{i=2^k}^{2^{k+1}-1} T^i f(x) \right) = \frac{1}{n+1} \left(f(x) + \sum_{k=0}^{j} 2^k \tilde{A}_{2^k} f(x) \right).
\end{aligned}$$

Thus

$$\begin{aligned}
\|A_{n,T}f\|_{L^p(wd\mu)} &\leq \frac{1}{n+1}\left(\|f\|_{L^p(wd\mu)} + \sum_{k=0}^{j} 2^k \|\tilde{A}_{2^k} f\|_{L^p(wd\mu)}\right) \\
&\leq \frac{1}{n+1}\left(\|f\|_{L^p(wd\mu)} + 2\,[w]^{1/p}_{A_p^+(T)}\|f\|_{L^p(wd\mu)} \sum_{k=0}^{j} 2^k\right) \\
&\leq \frac{1+2(2^{j+1}-1)}{n+1}\,[w]^{1/p}_{A_p^+(T)}\,\|f\|_{L^p(wd\mu)} \\
&= \frac{2^{j+2}-1}{n+1}\,[w]^{1/p}_{A_p^+(T)}\,\|f\|_{L^p(wd\mu)} \\
&\leq 4\,[w]^{1/p}_{A_p^+(T)}\,\|f\|_{L^p(wd\mu)}.
\end{aligned}$$

where we have used that $[w]^{1/p}_{A_p^+(T)} \geq 1$.

Now we prove the converse: if $\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)} < \infty$, then $w \in A_p^+(T)$ and

$$\frac{1}{2}[w]^{1/p}_{A_p^+(T)} \leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)}.$$

More precisely, we prove that for a.e. $x \in X$ and all $k \in \mathbb{N}$

$$\left(\sum_{i=-k}^{0} h_i^{-p}(x)J_i(x)\Phi^i w(x)\right)\left(\sum_{i=0}^{k}[h_i^{-p}(x)J_i(x)\Phi^i w(x)]^{\frac{-1}{p-1}}\right)^{p-1} \leq 2^p \sup_{n\in\mathbb{N}}\|A_{n,T}\|_{L^p(wd\mu)}(k+1)^p. \tag{4.2}$$

We start proving the following remark.

Remark 4.1. *Let $A = \{x : \Phi^i w(x) = 0\}$. For a.e $x \in A$, $\Phi^{i-j}w(x) = 0$ for all $j \geq 0$.*

Proof of 4.1. Since T is Cesàro bounded we have that

$$\begin{aligned}
||T^j(\Phi^{-i}\chi_A)||_{L^p(wd\mu)} &\leq (j+1)(\sup_{n\in\mathbb{N}}\|A_{n,T}\|_{L^p(wd\mu)})||\Phi^{-i}\chi_A||_{L^p(wd\mu)} \\
&= (j+1)(\sup_{n\in\mathbb{N}}\|A_{n,T}\|_{L^p(wd\mu)})\left(\int_X \Phi^{-i}\chi_A w\,d\mu\right)^{1/p} \\
&= (j+1)(\sup_{n\in\mathbb{N}}\|A_{n,T}\|_{L^p(wd\mu)})\left(\int_X J_i\chi_A \Phi^i w\,d\mu\right)^{1/p} = 0.
\end{aligned}$$

Thus $h_j(x)\Phi^{j-i}(\chi_A)(x)w(x) = 0$ a.e. Then $\Phi^{i-j}(h_{-i})(x)\chi_A(x)\Phi^{i-j}w(x) = 0$ a.e. and it follows that $\Phi^{i-j}w(x) = 0$ for a.e. $x \in A$. $\square$

Now we begin the proof of 4.2. Let us fix k. Let

$$Y = \{x : \sum_{i=0}^{k}[h_i^{-p}(x)J_i(x)\Phi^i w(x)]^{\frac{-1}{p-1}} = \infty\} = \cup_{i=0}^{k}\{x : \Phi^i w(x) = 0\}.$$

By Remark 4.1, we have that for a.e $x \in Y$, we have that $\Phi^i w(x) = 0$ for all $i \leq 0$. Therefore, (4.2) holds for a.e. $x \in Y$. Now, let

$$Z = \{x : \sum_{i=0}^{k}[h_i^{-p}(x)J_i(x)\Phi^i w(x)]^{\frac{-1}{p-1}} < \infty\}.$$

We shall prove that (4.2) holds for a.e. $x \in Z$. This completes the proof of (4.2) for a.e. $x \in X$.

As in the proof of the Lemma in [9] (see also [10]), we may assume without loss of generality that there exists an invertible measurable map $S : X \to X$ such that S^{-1} is measurable and $\Phi^j f = f \circ S^j$ for every $j \in \mathbb{Z}$ and all $f \in \mathcal{M}(\mu)$. Since Φ has no periodic part, for fixed $k \geq 0$, there exist sets B_j such that

$$Z = \bigcup_{j=0}^{\infty} B_j,$$

where the sets B_j satisfy the following:

$$B_j \cap S^l B_j = \emptyset \quad \text{for all } l \text{ such that } 1 \leq l \leq 2k.$$

Let us fix B_j y let A be any measurable subset of B_j with $0 < \mu(A) < \infty$. Let f be the function defined on X by

$$f(S^i x) = \begin{cases} h_i^{p'-1}(x)[J_i(x)w(S^i x)]^{\frac{-1}{p-1}} & \text{if } x \in A \text{ and } 0 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

Using the definition of f it follows that for $x \in A$ and $0 \le j \le k$ we have

$$A_{2k+1,T} f(S^{-j}x) = \frac{1}{2(k+1)} \sum_{i=0}^{2k+1} h_i(S^{-j}x) f[S^i(S^{-j}x)] = \frac{1}{2(k+1)} \sum_{i=j}^{k+j} h_i(S^{-j}x) f(S^{i-j}x)$$

$$= \frac{1}{2(k+1)} \sum_{i=0}^{k} h_{i+j}(S^{-j}x) f(S^i x) = \frac{1}{2(k+1)} \sum_{i=0}^{k} h_j(S^{-j}x) h_i(x) f(S^i x)$$

$$= \frac{1}{2(k+1)} h_j(S^{-j}x) \sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}}$$

$$= \frac{1}{2(k+1)} [h_{-j}(x)]^{-1} \sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}},$$

where in the last inequality we have used that $h_j(S^{-j}x) = [h_{-j}(x)]^{-1}$.

By property (2.3)

(4.3)

$$\int_{\bigcup_{j=0}^k S^{-j}A} |A_{2k+1,T} f(x)|^p w(x) d\mu = \sum_{j=0}^{k} \int_X |A_{2k+1,T} f(x)|^p \chi_{S^{-j}A}(x) w(x) d\mu$$

$$= \sum_{j=0}^{k} \int_X |A_{2k+1,T} f(S^{-j}x)|^p \chi_{S^{-j}A}(S^{-j}x) w(S^{-j}x) J_{-j}(x) d\mu$$

$$= \sum_{j=0}^{k} \int_A |A_{2k+1,T} f(S^{-j}x)|^p w(S^{-j}x) J_{-j}(x) d\mu$$

$$= \frac{1}{2^p(k+1)^p} \sum_{j=0}^{k} \int_A \Bigg[h_{-j}^{-p}(x) w(S^{-j}x) J_{-j}(x) \times \left(\sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}} \right)^p \Bigg] d\mu$$

$$= \frac{1}{2^p(k+1)^p} \int_A \left(\sum_{j=0}^{k} h_{-j}^{-p}(x) w(S^{-j}x) J_{-j}(x) \right) \times \left(\sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}} \right)^p d\mu.$$

Using the hypothesis, the fact that f is supported in $\cup_{i=0}^{k} S^i A$ and (2.3) we get

$$\begin{aligned}
\int_{\bigcup_{j=0}^{k} S^{-j} A} |A_{2k+1,T} f(x)|^p w(x) d\mu &\leq \|A_{2k+1,T}\|_p^p \int_{\bigcup_{i=0}^{k} S^{i} A} |f(x)|^p w(x) d\mu \\
&\leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} \sum_{i=0}^{k} \int_X |f(x)|^p \chi_{S^iA}(x) w(x) d\mu \\
&\leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} \sum_{i=0}^{k} \int_X |f(S^i x)|^p \chi_{S^iA}(S^i x) w(S^i x) J_i(x) d\mu \\
&\leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} \sum_{i=0}^{k} \int_A h_i^{p(p'-1)}(x) [J_i(x) w(S^i x)]^{\frac{-p}{p-1}} w(S^i x) J_i(x) d\mu \\
&\leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} \sum_{i=0}^{k} \int_A h_i^{p'}(x) [J_i(x) w(S^i x)]^{\frac{-1}{p-1}} d\mu \\
&\leq \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} \int_A \sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}} d\mu.
\end{aligned} \tag{4.4}$$

Putting together (4.3) and (4.4) we obtain

$$\begin{aligned}
&\int_A \left(\sum_{j=0}^{k} h_{-j}^{-p}(x) J_{-j}(x) w(S^{-j}x) \right) \left(\sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}} \right)^p d\mu \\
&\qquad \leq 2^p \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} (k+1)^p \int_A \sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}} d\mu.
\end{aligned}$$

Since A is any measurable subset of $B_j \subset Z$ with finite and positive measure, it follows that for all j and for a.e. $x \in B_j$ and, therefore, for a.e. $x \in Z$

$$\begin{aligned}
&\left(\sum_{j=0}^{k} h_{-j}^{-p}(x) J_{-j}(x) w(S^{-j}x) \right) \left(\sum_{i=0}^{k} [h_i^{-p}(x) J_i(x) w(S^i x)]^{\frac{-1}{p-1}} \right)^{p-1} \\
&\qquad\qquad \leq 2^p \sup_{n\in\mathbb{N}} \|A_{n,T}\|^p_{L^p(wd\mu)} (k+1)^p,
\end{aligned}$$

as we wished to prove. □

5. Proof of Theorem 3.1

As usual, the proof folllows by transference arguments from a result in the integers. We start with some definitions and the result we need on the integers.

If $f : \mathbb{Z} \to \mathbb{R}$ is any function then the one-sided maximal function m^+f on the integers is defined as follows:

$$m^+f(i) = \sup_{n\geq 0} \frac{1}{n+1} \sum_{j=0}^{n} \left|f(i+j)\right| = \sup_{n\geq 0} \frac{1}{n+1} \sum_{j=i}^{i+n} \left|f(j)\right|.$$

We point out that $m^+ = M_T$, where $Tf(i) = f(i+1)$. It is said that a weight w defined on $\mathbb{Z}$ belongs to $A_p^+(\mathbb{Z})$ if it is a nonnegative function such that

$$[w]_{A_p^+(\mathbb{Z})} := \sup_{j,k\in\mathbb{Z},k\geq 0} \left(\frac{1}{k+1} \sum_{i=j-k}^{j} w(i)\right) \left(\frac{1}{k+1} \sum_{i=j}^{j+k} w(i)^{\frac{-1}{p-1}}\right)^{p-1} < +\infty. \tag{5.1}$$

The quantity $[w]_{A_p^+(\mathbb{Z})}$ is known as the characteristic of the weight w.

It is well known that if $w \in A_p^+(\mathbb{Z})$ then there exists $C \geq 0$ such that

$$\left(\sum_{i=\infty}^{\infty} |m^+f(i)|^p w(i)\right)^{1/p} \leq C \left(\sum_{i=\infty}^{\infty} |f(i)|^p w(i)\right)^{1/p}. \tag{5.2}$$

for all $f \in L^p(\mathbb{Z}, w)$. As usual, the least constant C in (5.2) is the norm of m^+ and it is denoted by $\|m^+\|_{L^p(\mathbb{Z},w)}$. The next theorem follows from the results in [8] and gives the sharp constant in the above inequality.

Theorem 5.1. *Let w be a weight defined on $\mathbb{Z}$ and let $1 < p < \infty$. If $w \in A_p^+(\mathbb{Z})$ then there exists a constant $C(p)$ such that*

$$\|m^+\|_{L^p(\mathbb{Z},w)} \leq C(p)[w]_{A_p^+(\mathbb{Z})}^{\frac{1}{p-1}}.$$

Furthermore, the exponent is sharp, that is, if $\beta \geq 0$ and $C(p)$ is a constant such that $\|m^+\|_{L^p(\mathbb{Z},w)} \leq C(p)[w]^{\beta}_{A_p^+(\mathbb{Z})}$ for all $w \in A_p^+(\mathbb{Z})$, then $\beta \geq \frac{1}{p-1}$.

Although the proof follows from the results in [8], for reasons of completeness, we give an sketch of the proof of this result in Section 8.

5.1. **Proof of Theorem 3.1.** For fixed $x \in X$, let $u^x(i) = h_i^{-p}(x)J_i(x)\Phi^i w(x)$ a function defined on the integers. By Theorem 3.3 and Remark 3.5 we have that for a.e. $x \in X$ the functions u^x belong to $A_p^+(\mathbb{Z})$ and

$$[u^x]_{A_p^+(\mathbb{Z})} \leq 2^p(\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)})^p$$

for a.e. $x \in X$.

Now we start the proof of the boundedness of M_T. It is enough to work with non-negative measurable functions f. For any natural number L, we consider the truncated maximal operator

$$M_{T,L}f = \sup_{0\leq n\leq L} A_{n,T}f. \tag{5.3}$$

Let N be any natural number. By (2.3), we have

$$\int_X (M_{T,L}f)^p w\, d\mu = \frac{1}{N+1}\int_X \sum_{i=0}^{N} (\Phi^i(M_{T,L}f))^p \Phi^i w J_i\, d\mu. \tag{5.4}$$

Let f^x the function on the integers given by $f^x(i) = T^i f(x)$ and let $[0, N+L]$ be the interval $\{0, 1, \ldots, N+L\}$. By the properties of the functions h_j we have

$$\Phi^i(M_{T,L}f)(x) \leq (h_i(x))^{-1} m^+(f^x\chi_{[0,N+L]})(i).$$

Then

$$\begin{aligned}\sum_{i=0}^{N}(\Phi^i(M_{T,L}f))^p(x)\Phi^i w(x)J_i(x) &\leq \sum_{i=0}^{N}(m^+(f^x\chi_{[0,N+L]})^p(i)(h_i(x))^{-p}J_i(x)\Phi^i w(x)\\ &\leq \sum_{i=-\infty}^{\infty}(m^+(f^x\chi_{[0,N+L]})^p(i)u^x(i),\end{aligned}$$

where, as before, $u^x(i) = (h_i(x))^{-p} J_i(x)\Phi^i w(x)$. By Theorem 5.1, for a.e. $x \in X$

$$\sum_{i=0}^{N} (\Phi^i(M_{T,L}f))^p(x)\Phi^i w(x) J_i(x) \le C(p)[u^x]^{p'}_{A_p^+(\mathbb{Z})} \sum_{i=0}^{N+L} (f^x)^p(i) u^x(i)$$
$$\le C(p) 2^{pp'} (\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)})^{pp'} \sum_{i=0}^{N+L} \Phi^i f^p(x) J_i(x) \Phi^i w(x).$$

The last inequality together with (5.4) gives

$$\int_X (M_{T,L}f)^p w\, d\mu \le C(p)2^{pp'} (\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)})^{pp'} \frac{1}{N+1} \int_X \sum_{i=0}^{N+L} \Phi^i f^p J_i \Phi^i w\, d\mu$$
$$= C(p)2^{pp'} (\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)})^{pp'} \frac{N+L+1}{N+1} \int_X f^p w\, d\mu.$$

Taking limit as $N \to \infty$,

$$\int_X (M_{T,L}f)^p w\, d\mu \le C(p)2^{pp'} (\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)})^{pp'} \int_X f^p w\, d\mu.$$

Finally, letting L go to ∞,

$$\int_X (M_T f)^p w\, d\mu \le C(p)2^{pp'} (\sup_{n\in\mathbb{N}} \|A_{n,T}\|_{L^p(wd\mu)})^{pp'} \int_X f^p w\, d\mu,$$

as we wished to prove.

6. Proof of Theorem 3.6

Proof. It is well known that M_{T_p} is bounded in $L^p(d\mu)$ and

$$\|M_{T_p}\|_{L^p(d\mu)} \le \frac{p}{p-1},$$

(see [2]). In what follows, we shall prove

$$\frac{p}{p-1} \le \|M_{T_p}\|_{L^p(d\mu)}.$$

As before, we may assume, without loss of generality, that there exists an invertible measurable map $S : X \to X$ such that S^{-1} is measurable and $\Phi^j f = f \circ S^j$ for every $j \in \mathbb{Z}$ and all $f \in \mathcal{M}(\mu)$. Also, as before, since Φ has no periodic part, for all natural numbers k there exist measurable sets B_j such that

$$X = \bigcup_{j=0}^{\infty} B_j \quad \text{and} \quad B_j \cap S^l B_j = \emptyset, \quad 1 \leq l \leq 2k.$$

Let us fix a measurable subset $A \subset B_0$ such that $0 < \mu(A) < \infty$ and consider the function

$$f(x) = \sum_{j=0}^{k} \frac{1}{(j+1)^{1/p}} J_j(x)^{1/p} \chi_{S^{-j}A}(x).$$

Let $0 \leq i \leq k$ and $x \in S^{-i}A$. It follows from the definition of T_p and (2.3) that for all l, $0 \leq l \leq i$,

$$\begin{aligned}
T_p^l f(x) &= J_l(x)^{1/p} \Phi^l f(x) \\
&= J_l(x)^{1/p} \sum_{j=0}^{k} \frac{1}{(j+1)^{1/p}} J_j(S^l x)^{1/p} \chi_{S^{-j}A}(S^l x) \\
&= \frac{1}{(i-l+1)^{1/p}} J_l(x)^{1/p} J_{i-l}(S^l x)^{1/p} \\
&= \frac{1}{(i-l+1)^{1/p}} J_i(x)^{1/p}.
\end{aligned}$$

Therefore, if $x \in S^{-i}A$ then

$$M_{T_p} f(x) \geq \frac{1}{i+1} \sum_{l=0}^{i} T_p^l f(x) = \frac{J_i(x)^{1/p}}{i+1} \sum_{l=0}^{i} \frac{1}{(i-l+1)^{1/p}} = \frac{J_i(x)^{1/p}}{i+1} \sum_{j=0}^{i} \frac{1}{(j+1)^{1/p}}.$$

Thus

$$\begin{aligned}
\int_{\bigcup_{i=0}^k S^{-i}A} |M_{T_p}f(x)|^p d\mu &= \sum_{i=0}^k \int_{S^{-i}A} |M_{T_p}f(x)|^p d\mu \\
&\geq \sum_{i=0}^k \int_{S^{-i}A} \left| \frac{J_i(x)^{1/p}}{i+1} \sum_{j=0}^i \frac{1}{(j+1)^{1/p}} \right|^p d\mu \\
&= \sum_{i=0}^k \left(\frac{1}{i+1} \sum_{j=0}^i \frac{1}{(j+1)^{1/p}} \right)^p \int_X J_i(x)\chi_{S^{-i}A}(x) d\mu \\
&= \sum_{i=0}^k \left(\frac{1}{i+1} \sum_{j=0}^i \frac{1}{(j+1)^{1/p}} \right)^p \int_X J_i(x)\chi_A(S^i x) d\mu \\
&= \mu(A) \sum_{i=0}^k \left(\frac{1}{i+1} \sum_{j=0}^i \frac{1}{(j+1)^{1/p}} \right)^p .
\end{aligned}$$

Now we apply that M_{T_p} is bounded in $L^p(d\mu)$ and we obtain

$$\begin{aligned}
\int_{\bigcup_{i=0}^k S^{-i}A} |M_{T_p}f(x)|^p d\mu &\leq \|M_{T_p}\|_{L^p(d\mu)}^p \int_X |f(x)|^p d\mu \\
&= \|M_{T_p}\|_{L^p(d\mu)}^p \sum_{j=0}^k \frac{1}{j+1} \int_X \chi_{S^{-j}A}(x) J_j(x)\, d\mu \\
&= \|M_{T_p}\|_{L^p(d\mu)}^p \sum_{j=0}^k \frac{1}{j+1} \int_X \chi_A(S^j x) J_j(x)\, d\mu \\
&= \|M_{T_p}\|_{L^p(d\mu)}^p \mu(A) \sum_{j=0}^k \frac{1}{j+1} .
\end{aligned}$$

Putting together both inequalities we have

$$\frac{\sum_{i=0}^k \left(\frac{1}{i+1} \sum_{j=0}^i \frac{1}{(j+1)^{1/p}} \right)^p}{\sum_{j=0}^k \frac{1}{j+1}} \leq \|M_{T_p}\|_{L^p(d\mu)}^p . \tag{6.1}$$

We compute the limit of the sequence on the left hand side by applying Stolz-Cesàro theorem. We consider the sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ where

$$a_k=\sum_{i=0}^{k}\left(\frac{1}{i+1}\sum_{j=0}^{i}\frac{1}{(j+1)^{1/p}}\right)^p \qquad\text{and}\qquad b_k=\sum_{j=0}^{k}\frac{1}{j+1}.$$

It is easy to see that

$$\frac{a_k-a_{k-1}}{b_k-b_{k-1}}=\frac{\left(\frac{1}{k+1}\sum_{j=0}^{k}\frac{1}{(j+1)^{1/p}}\right)^p}{\frac{1}{k+1}}=\left(\frac{\sum_{j=1}^{k+1}\frac{1}{j^{1/p}}}{(k+1)^{1-\frac{1}{p}}}\right)^p.$$

We observe that the term into the brackets is a Riemann sum of the function $x^{-1/p}$ on the interval $[0,1]$. Taking limit and applying Stolz-Cesàro theorem we obtain

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\left(\frac{\sum_{j=1}^{k+1}\frac{1}{j^{1/p}}}{(k+1)^{1-\frac{1}{p}}}\right)^p=\left(\int_0^1 x^{-1/p}\right)^p=\left(\frac{p}{p-1}\right)^p.$$

This limit together with (6.1) gives

$$\frac{p}{p-1}\le\|M_{T_p}\|_{L^p(d\mu)}.$$

□

7. Proof of Theorem 3.2

We start with the following lemma which is interesting by itself.

Lemma 7.1. *Let $1<p<p_0<\infty$ and let T_p a positive invertible isometry on $L^p(d\mu)$, $T_pf=J_1^{1/p}\Phi f$, such that Φ has no periodic part. For each $f\in L^p(d\mu)$ let*

$$R_pf=\sum_{k=0}^{\infty}\frac{M_p^kf}{(2p')^k},$$

where $M_p=M_{T_p}$ is the ergodic maximal operator associated to T_p, $M_p^0f=f$, $M_p^{k+1}f=M_p(M_p^kf)$ and $p+p'=pp'$. Finally, let $w=(R_pf)^{p-p_0}$. Then T_p is Cesàro bounded in

$L^{p_0}(wd\mu)$ *and*

$$\sup_{n\in\mathbb{N}} \|A_{n,T_p}\|_{L^{p_0}(wd\mu)} \le 4(4p')^{(p_0-p)/p_0}. \tag{7.1}$$

Proof of Lemma 7.1. We recall that the maximal operator M_p is bounded on $L^p(d\mu)$ and $||M_p||_{L^p(d\mu)} = p/(p-1) = p'$. Then it is clear that

$$R_pf \in L^p(d\mu), \quad |f| \le R_pf, \quad ||R_pf||_{L^p(d\mu)} \le 2||f||_{L^p(d\mu)} \quad \text{and} \quad M_p(R_pf) \le 2p'R_pf. \tag{7.2}$$

It follows from the last inequality that if $k \ge 0$ and $-k \le i \le 0$ then

$$\frac{1}{k+1}\sum_{j=0}^{k} T_p^j(R_pf) \le 4p'T_p^i(R_pf) \quad \text{a.e. } x. \tag{7.3}$$

Notice that this property implies that, for a.e. x, if $T_p^i(R_pf)(x) = 0$ for some i, $-k \le i \le 0$, then $T_p^j(R_pf) = 0$ for $0 \le j \le k$ (in fact for all $j \ge i$). Taking into account this remark it follows from (7.3) that

$$\sum_{i=-k}^{0} (T_p^i(R_pf))^{1-p_0}\left(\sum_{i=0}^{k} T_p^i(R_pf)\right)^{p_0-1} \le (4p')^{p_0-1}(k+1)^{p_0} \quad \text{a.e. } x. \tag{7.4}$$

Now we proceed to prove that T_p is Cesàro bounded in $L^{p_0}(wd\mu)$. By Theorem 3.3, it suffices to prove that $w \in A^+_{p_0}(T_p)$. More precisely, we will prove that for a.e. $x \in X$ and all $k \in \mathbb{N}$

$$\left(\sum_{i=-k}^{0} J_i^{-p_0/p}(x)J_i(x)\Phi^i w(x)\right)\left(\sum_{i=0}^{k}[J_i^{-p_0/p}(x)J_i(x)\Phi^i w(x)]^{\frac{-1}{p_0-1}}\right)^{p_0-1} \le (4p')^{p_0-p}(k+1)^{p_0}. \tag{7.5}$$

By Hölder's inequality with exponents $q = \frac{p_0-1}{p_0-p}$ and $q' = \frac{p_0-1}{p-1}$ applied to both sums in (7.5) we get that the left hand side of (7.5) is bounded by

$$\left(\sum_{i=-k}^{0} (T_p^i(R_pf))^{1-p_0}\right)^{\frac{p_0-p}{p_0-1}} (k+1)^{\frac{p-1}{p_0-1}} \left(\sum_{i=0}^{k} T_p^i(R_pf)\right)^{(p_0-1)\frac{p_0-p}{p_0-1}} (k+1)^{\frac{p-1}{p_0-1}(p_0-1)} \quad \text{a.e. } x. \tag{7.6}$$

Using (7.4) we obtain (7.5) and the lemma is completely proved since (7.1) follows from (7.5) and Theorem 3.3. □

Proof of Theorem 3.2. We follow in this proof the ideas in [6].

Let $f \in L^p(d\mu)$ and let $M_p f$, $R_p f$ and w be as in Lemma 7.1. Applying Hölder's inequality with exponent p_0/p we obtain

$$\begin{aligned}\|M_p f\|_{L^p(d\mu)} &= \left(\int_X |M_p f|^p (R_p f)^{(p-p_0)\frac{p}{p_0}} (R_p f)^{(p_0-p)\frac{p}{p_0}}\, d\mu\right)^{\frac{1}{p}} \\ &\le \left(\int_X |M_p f|^{p_0} w\, d\mu\right)^{\frac{1}{p_0}} \left(\int_X (R_p f)^p\, d\mu\right)^{\frac{p_0-p}{p_0 p}}.\end{aligned} \tag{7.7}$$

By Lemma 7.1, T_p is Cesàro bounded in $L^{p_0}(wd\mu)$ and (7.1) holds. Then, by the assumption of Theorem 3.2,

$$\begin{aligned}\left(\int_X |M_p f|^{p_0} w\, d\mu\right)^{\frac{1}{p_0}} &\le C(p_0)(4(4p')^{(p_0-p)/p_0})^{\beta} \left(\int_X |f|^{p_0} (R_p f)^{p-p_0}\, d\mu\right)^{\frac{1}{p_0}} \\ &\le C(p_0)(4(4p')^{(p_0-p)/p_0})^{\beta} \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where in the last inequality we have used that $|f| \le R_p(f)$ (see (7.2)).

By (7.2)

$$\left(\int_X (R_p f)^p\, d\mu\right)^{\frac{p_0-p}{p_0 p}} \le 2^{\frac{p_0-p}{p_0}} \left(\int_X |f|^p\, d\mu\right)^{\frac{p_0-p}{p_0 p}}.$$

The last inequalities together with (7.7) give

$$\|M_p f\|_{L^p(d\mu)} \le C(p_0) 2^{\frac{p_0-p}{p_0}} (4(4p')^{(p_0-p)/p_0})^{\beta} \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}}.$$

Since $\|M_p\|_{L^p(d\mu)} = p/(p-1) = p'$,

$$p' \le C(p_0) 2^{\frac{p_0-p}{p_0}} (4(4p')^{(p_0-p)/p_0})^{\beta}.$$

Taking limit as p goes to 1, we obtain that

$$1 \leq \frac{p_0 - 1}{p_0}\beta$$

or, in other words $\beta \geq p_0'$, as we wished to prove. □

8. Sketch of the proof of Theorem 5.1

We recall notations and results in [8].

Let μ be a Borel measure on the real line which is finite on bounded sets. For any measurable function F on the real line we define the one-sided maximal functions

$$M_\mu^+ F(x) = \sup_{h>0} \frac{1}{\mu([x,x+h))} \int_{[x,x+h)} |F|\, d\mu,$$

and

$$M_\mu^- F(x) = \sup_{h>0} \frac{1}{\mu((x-h,x])} \int_{(x-h,x]} |F|\, d\mu,$$

where the respective quotients are understood as zero when $\mu([x,x+h)) = 0$ or $\mu((x-h,x]) = 0$. We also introduce the following notations: given real numbers $a \leq b \leq c$, $\{a,b]$ and $[b,c\}$ will stand for $(a,b]$ or $[a,b]$ and $[b,c)$ or $[b,c]$, respectively, while $\{a,c\}$ will denote the union $\{a,b] \cup [b,c\}$.

Definition 8.1. *Let $1 < p < \infty$. Let W be a weight on the real line (a nonnegative measurable function). The one-sided constant $[W]_{A_p^+(\mu)}$ is defined as*

$$[W]_{A_p^+(\mu)} := \sup_{(a,b,c)\in\mathcal{T}} \left(\frac{1}{\mu(\{a,b])} \int_{\{a,b]} W\, d\mu\right) \left(\frac{1}{\mu([b,c\})} \int_{[b,c\}} W^{1-p'}\, d\mu\right)^{p-1}, \tag{8.1}$$

where the supremum is taken over the set $\mathcal{T}$ of triplets (a,b,c) such that

$$\mu(\{a,c\}) > 0, \quad \mu(\{a,b]) \geq \frac{1}{2}\mu(\{a,c\}) \quad \textit{and} \quad \mu([b,c\}) \geq \frac{1}{2}\mu(\{a,c\}).$$

The one-sided constant $[W]_{A_p^-(\mu)}$ *is defined reversing the orientation of the real line:*

$$(8.2) \qquad [W]_{A_p^-(\mu)} := \sup_{(a,b,c)\in\mathcal{T}} \left(\frac{1}{\mu([b,c\})}\int_{[b,c\}} W\,d\mu\right)\left(\frac{1}{\mu(\{a,b]))}\int_{\{a,b]} W^{1-p'}\,d\mu\right)^{p-1}.$$

Theorem 8.2. *[([8]), Buckley's theorem for one-sided maximal operators] Let* $1 < p < +\infty$*. Let* W *be a weight in* $\mathbb{R}$*. The following assertions are equivalent.*

(a) $[W]_{A_p^+(\mu)} < +\infty$.

(b) M_μ^+ *is bounded on* $L^p(W d\mu)$.

Moreover, if any of the above conditions hold then

$$\frac{1}{2}[W]^{\frac{1}{p}}_{A_p^+(\mu)} \le ||M_\mu^+||_{\mathcal{B}(L^p(W d\mu))} \le 2ep'[W]^{\frac{1}{p-1}}_{A_p^+(\mu)}.$$

Proof of Theorem 5.1. Let μ be the measure on the real line defined as the sum of the Dirac deltas on the integers. For any real number x, let $[x]$ be the integer part of x. Given any function f on the integers, let F be the function on the real line defined as $F(x) = f([x])$. Taking into account this notation, we have the following two lemmas.

Lemma 8.3. *Let* μ *be the measure on the real line defined as the sum of the Dirac deltas on the integers. The weight* $w \in A_p^+(\mathbb{Z})$ *if and only if* $W(x) = w([x]) \in A_p^+(\mu)$*. Furthermore, there exists a constant* $C(p)$ *such that*

$$[w]_{A_p^+(\mathbb{Z})} \le [W]_{A_p^+(\mu)} \le C(p)[w]_{A_p^+(\mathbb{Z})}.$$

Lemma 8.4. *For any function* f *on the integers and all* $j \in \mathbb{Z}$*, we have*

$$m^+ f(j) = M_\mu^+ F(j),$$

with $F(x) = f([x])$.

The proofs of both lemmas are quite direct. So we left to the reader to fill the details of the proofs.

It follows from Lemmas 8.3 and 8.4 that

$$\|m^+ f\|_{L^p(\mathbb{Z},w)} = \|M_\mu^+ F\|_{L^p(W\,d\mu)} \leq 2ep'[W]_{A_p^+(\mu)}^{\frac{1}{p-1}} \leq 2ep'(C(p)[w]_{A_p^+(\mathbb{Z})})^{\frac{1}{p-1}},$$

as we wished to prove. □

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